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Small-Size FDCT/IDCT Algorithms with Reduced Multiplicative Complexity

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Abstract—Discrete orthogonal transforms including the discrete Fourier transform, the discrete Walsh transform, the discrete Hartley transform, the discrete Slant transform, etc. are extensively used in radio-electronic and telecommunication systems for data processing and transmission. The popularity of using these transform is explained by the presence of fast algorithms that minimize the computational and hardware complexity of their implementation. A special place in the list of transforms is occupied by the forward and inverse discrete cosine transforms (FDCT and IDCT respectively). This article proposes a set of parallel algorithms for the fast implementation of FDCT/IDCT. The effectiveness of the proposed solutions is justified by the possibility of the factorization of the FDCT/IDCT matrices, which leads to a decrease in computational and implementation complexity. Some fully parallel FDCT/IDCT algorithms for small lengths $N = 2, 3, 4, 5, 6, 7$ are presented.

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1. INTRODUCTION

Traditional types of discrete orthogonal transforms such as discrete Fourier transform, discrete Hartley transform, discrete Walsh and Haar transform, Slant transform, discrete cosine transform (DCT) are important data processing tools [1]. In order to minimize the computational complexity and hardware resources for implementing these transforms, various fast algorithms have been developed [2–4]. Among the arsenal of the above transformations, DCT is one of the most important [5–9].

DCT has found wide applications in many scientific and technological fields, including data compression [10, 11], digital signal and image processing [12–14], digital television [15–18], digital watermarking [19–23], telecommunications [24, 25] and others. Fast algorithms for this type of transform were described in [1–9, 26–54].

Most of these publications are devoted to DCT algorithms for sequences of length 8, since transforms of data blocks of precisely these sizes are used in data compression standards [26–31]. A part of the known papers is devoted to one-dimensional and two-dimensional DCT algorithms oriented to processing of data sequences whose length is a power of two [32–52], another part of the publications deals with the so-called “prime factor algorithms” [51–54].

The laboriousness and time of calculating the DCT was the reason for the appearance of a number of publications devoted to the development of hardware accelerators of this transform, implemented on VLSI platforms: ASIC and FPGA [55–76].

In some cases, the fast algorithms for discrete orthogonal transforms for short-lengths input sequences are of practical interest [77–80]. In the case of the hardware implementation of digital signal and image processing techniques using DCT, the small-size transforms kernels can serve as building blocks for the implementation of more complex algorithms, such as modified discrete cosine transform, overlapping orthogonal transform, cosine-modulated filter banks, discrete wavelet-like transform [81–83].

However, the DCT algorithms for short-length sequences are described in publications accessible to authors rather poorly. Information concerning small-size DCT transforms is scattered, and the solutions presented in the works known to the authors are not always accurate. As a rule, authors of well-known

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